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THE EFFECT OF SHRINKAGE ON THE RATE OF SOLIDIFICATION OF A CYLINDRICAL INGOT

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Abstract — As it cools the solidified part of an ingot contracts and creates a small gap between the ingot and its mould. Equations are formulated which account for the existence of such a gap in which the heat transfer is presumed purely radiative. Numerical solutions show that the presence of the gap increases the time of complete solidification and accounts for previous underestimates of this time as compared with observations in the steel industry.

NOMENCLATURE

a, inner radius of mould;
b, outer radius of mould;
C_p, specific heat;
k, thermal conductivity;
r, radial co-ordinate;
t, time;
t_s, time of complete solidification;
Bi, Biot number;
E, interface position;
L, latent heat of fusion;
T, temperature;
T_e, environmental temperature;
T_c^*, preheat temperature;
T_f, fusion temperature;
α, ratio of emissivities, $\alpha = \varepsilon_1/\varepsilon_0 = \varepsilon_2/\varepsilon_0$;
H, $= 1 - (a/b)$;
β, $= L/C_p\Delta T$;
γ, heat-transfer coefficient;
ΔT, $\Delta T = T_f - T_e$;
$\varepsilon(\tau)$, non-dimensional penetration;
ε_0, emissivity of outer surface of mould;
ε_1, emissivity of inner surface of mould;
ε_2, emissivity of ingot surface;
ξ, dimensionless co-ordinate;
η, dimensionless co-ordinate;
κ, thermal diffusivity;
ρ, density;
σ, Stefan–Boltzmann constant;
τ, dimensionless time, $\tau = t\kappa/a^2$;
θ, dimensionless temperature.

Superscript

*, relates to a mould quantity.

INTRODUCTION

ABUNDANT interest in solidification problems exists because of the wide range of physical processes in which the phenomenon is important. Of particular interest is the manner in which ingots solidify in iron, steel and brass foundries. Ingots are often modelled by infinite cylinders for which the heat-transfer equation has been solved subject to a variety of boundary conditions. The simplest case when the outer surface is maintained at a prescribed constant temperature has been the subject of considerable attention, e.g. [1]–[4], but the continued interest is largely due to the existence of a mathematical singularity at the final time of solidification (finally resolved by Stewartson and Waechter [4]) rather than any practical use. More interesting physically is the work of Goodling and Khader [5] who imposed a combination of Newton cooling and Stefan's law at the boundary and solved their equations numerically. However although there is no way of making these boundary conditions more realistic the model underestimates the time of complete solidification compared with observations in the steel industry, see Ruddle [6]. An obvious simplifying feature as compared with foundry practice is the absence of a containing mould but inclusion of this, see [7], does not account for the underestimation. Another simplification in most of the work referred to so far is that the molten metal is presumed to be initially at its fusion temperature T_f whereas the molten metal must be poured at a higher temperature (typically 20°C in the case of steel), but this too adds only marginally to the time of complete solidification.

In this paper the effect of shrinkage is included in so far as it gives rise to a narrow gap between the outer surface of the solidifying ingot and the inner surface of the mould. The gap undoubtedly exists, see [6] and [8], being created as the solidified phase cools and contracts. There is also associated contraction of the mould (after perhaps initial expansion) but the greater heat loss and greater contraction occur in the ingot. The mechanism for the contraction does not enter the calculation because the heat transfer across the gap is presumed purely radiative and hence the precise gap width is not relevant when calculating the heat transfer. Instead a gap is assumed to exist, that the heat transfer is radiative between the ingot and inner mould surface but any changes in the overall dimensions are ignored when specifying where the boundary conditions apply.

THE GOVERNING EQUATIONS

The molten metal is initially at its fusion tempera-

ture T_f and contained in a sleeve mould with internal and external radii a and b respectively. The temperature within the mould at any instant of time t is denoted by $T^*(r,t)$, where r is a measure of distance from the axis of the mould, and initially this is prescribed as $T_i^*(r)$. The mould is situated in an environment at temperature $T_e(< T_f)$ so as time proceeds heat is drawn through the mould and the molten metal solidifies inwards; the location of the solid–liquid interface is denoted by $r = a - E(t)$, where $E(t)$ is the depth of penetration, and the temperature in the solidified phase by $T(r,t)$.

Heat transfer within the ingot and mould is due to conduction and thus $T(r,t)$ and $T^*(r,t)$ satisfy

$$\frac{\partial T}{\partial t} = \kappa \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \quad a - E(t) \leqslant r \leqslant a, \tag{1}$$

$$\frac{\partial T^*}{\partial t} = \kappa^* \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^*}{\partial r}\right), \quad a \leqslant r \leqslant b, \tag{2}$$

where κ and κ^* are the thermal diffusivities of the solidified metal and the mould respectively: in general starred quantities relate to the mould. These equations are solved subject to the following conditions.

At the outer surface of the mould heat transfer is modelled by a combination of Stefan radiation and Newton cooling,

$$-k^*\left(\frac{\partial T^*}{\partial r}\right) = \gamma(T^* - T_e) + \varepsilon_0\sigma(T^{*4} - T_e^4), \quad r = b, \tag{3}$$

where k^* is the thermal conductivity of the mould, γ is the heat-transfer coefficient, ε_0 the emissivity of the outer mould surface and σ the Stefan–Boltzman constant.

Heat is radiated between the surface of the ingot and the inner surface of the mould according to

$$-k\left(\frac{\partial T}{\partial r}\right) = \varepsilon_1\sigma T^4 - \varepsilon_2\sigma T^{*4} = -k^*\left(\frac{\partial T^*}{\partial r}\right), \quad r = a, \tag{4}$$

where k is the thermal conductivity of the metal, ε_1 and ε_2 are surface emissivities of the metal and the mould inner surface, and the gap is presumed to have negligible width so that the temperature gradients are both evaluated when $r = a$.

At the liquid/solid interface, continuity in temperature and liberation of latent heat yield

$$T = T_f, \quad r = a - E(t) \tag{5}$$

$$k\left(\frac{\partial T}{\partial r}\right) = -\rho L\frac{\mathrm{d}E}{\mathrm{d}t}, \quad r = a - E(t) \tag{6}$$

where ρ is the density of the metal (presumed constant) and L is the latent heat of fusion.

The system (1)–(6) is non-dimensionalized by introducing

$$\tau = \frac{\kappa t}{a^2}, \quad \varepsilon(\tau) = E(t)/a, \quad \xi = \frac{b - r}{b - a}, \quad \eta = \frac{a - r}{a\varepsilon(\tau)}, \quad \theta^* = \frac{T^* - T_e}{T_f - T_e}, \quad \theta = \frac{T - T_e}{T_f - T_e} \tag{7}$$

which have the effect of defining the mould by $0 \leqslant \xi \leqslant 1$ and the solidified part of the ingot by $0 \leqslant \eta \leqslant 1$. In terms of the new variables the problem is characterized by

$$\frac{\kappa}{\kappa^*}\frac{b^2H^2}{a^2}\frac{\partial\theta^*}{\partial\tau} = \frac{\partial^2\theta^*}{\partial\xi^2} - \frac{H}{1 - H\xi}\frac{\partial\theta^*}{\partial\xi}, \quad 0 \leqslant \xi \leqslant 1, \tag{8}$$

$$\varepsilon(\tau)^2\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial\eta^2} + \left(\eta\varepsilon(\tau)\dot{\varepsilon}(\tau) - \frac{\varepsilon(\tau)}{1 - \eta\varepsilon(\tau)}\right)\frac{\partial\theta}{\partial\eta}, \quad 0 \leqslant \eta \leqslant 1, \tag{9}$$

subject to the conditions

$$-\left(\frac{\partial\theta^*}{\partial\xi}\right)_{\xi=0} + Bi\theta^*_{\xi=0} + A_0\left\{\theta^{*4} + 4\frac{T_e}{\Delta T}\theta^{*3} + 6\left(\frac{T_e}{\Delta T}\right)^2\theta^{*2} + 4\left(\frac{T_e}{\Delta T}\right)^3\theta^*\right\}_{\xi=0} = 0 \tag{10}$$

$$\left(\frac{\partial\theta^*}{\partial\xi}\right)_{\xi=1} = A_1\left\{\theta^4 + 4\frac{T_e}{\Delta T}\theta^3 + 6\left(\frac{T_e}{\Delta T}\right)^2\theta^2 + 4\left(\frac{T_e}{\Delta T}\right)^3\theta + \left(\frac{T_e}{\Delta T}\right)^4\right\}_{\eta=0} - A_2\left\{\theta^{*4} + 4\frac{T_e}{\Delta T}\theta^{*3} + 6\left(\frac{T_e}{\Delta T}\right)^2\theta^{*2} + 4\left(\frac{T_e}{\Delta T}\right)^3\theta^* + \left(\frac{T_e}{\Delta T}\right)^4\right\}_{\xi=1}, \tag{11}$$

$$\frac{k}{k^*}\left(\frac{b}{a} - 1\right)\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = \varepsilon(\tau)\left(\frac{\partial\theta^*}{\partial\xi}\right)_{\xi=1}, \tag{12}$$

$$\theta = 1 \quad \text{at } \eta = 1, \tag{13}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} = \beta\varepsilon(\tau)\dot{\varepsilon}(\tau), \tag{14}$$

where $H = 1 - (a/b)$, $Bi = \gamma(b - a)/k^*$ is the Biot number, $\Delta T = T_f - T_e$, $A_i = (b - a)\sigma\varepsilon_i(\Delta T)^3/k^*$ are the radiation parameters ($i = 0, 1, 2$) and $\beta = L/C_p\Delta T$ is the inverse of the Stefan number.

METHOD OF SOLUTION

Assuming the position of the interface and thermal distributions ε_N, θ_N and θ_N^* are known at time $\tau = \tau_N$ the corresponding values at $\tau = \tau_{N+1} = \tau_N + \delta\tau$ are deduced by applying the Hartree–Womersley scheme [9]. In this method time derivatives are replaced by differences of the type $(\partial\theta/\partial\tau) = (\theta_{N+1} - \theta_N)/\delta\tau$ and all other quantities by averages. The resulting ordinary differential equations for θ_{N+1} and θ_{N+1}^* together with all the boundary conditions are linearized and the entire set of linear equations solved simultaneously as part of the inner iteration. Convergence of the inner

iteration gives the values of $\theta_{N+1}, \theta^*_{N+1}$ and ε_{N+1} which are then used in the outer iteration in place of θ_N, θ^*_N and ε_N. The inner iteration involves only two or three steps, while the number of outer iterations depends on the size of the time increment $\delta\tau$. The scheme is started when $\tau = 0$ and $\varepsilon = 0$ and continued until $\varepsilon = 1$ at which time the region $r < a$ is completely solidified. When $\varepsilon = 1$ a singularity can be noted in equation (9) so in practice the numerical scheme does not reach $\varepsilon = 1$ but by taking smaller and smaller time increments as $\varepsilon \to 1$ (necessary anyway to retain accuracy) any tolerance can be prescribed for the final time of solidification, τ_s.

THE INITIAL TEMPERATURE DISTRIBUTION IN THE MOULD

The preceding algorithm is complete for moving from the solution at $\tau = \tau_N$ to that at $\tau = \tau_{N+1}$; all that is required to completely determine the problem is the precise form of $\theta^*(\xi, 0)$, i.e. the non-dimensional form of $T^*_I(r)$. In practice a mould is often preheated to avoid supercooling of the first drops of molten metal poured into the mould; assuming the initial temperature of the inner mould surface is T^*_c an appropriate form of $T^*_I(r)$ is obtained by solving the steady state version of (2) subject to the outer wall condition (3) and the inner wall condition $T^*_I(a) = T^*_c$. The non-dimensional temperature then has the form

$$\theta^* = C\log(1 - H\xi) + D,$$

where C and D are constants obtained from the boundary conditions. Since $T^*_c < T_f$ there is an initial singularity (just as in the case of a constant outer temperature without a mould) and though this leads to a small numerical oscillation about the true solution this is a stable process which does not affect the time of complete solidification.

DISCUSSION OF RESULTS

Results are presented which are particularly relevant to a range of steel ingots in production today. Ingots are not perfectly cylindrical but rather more square in section with rounded corners; nevertheless a cylinder is an appropriate model and the dimensions shown in Tables 1 and 2 give the same cross-sectional areas as the moulds in common use. Having specified the thermal properties and the overall dimensions the remaining parameters which have an effect on the solidification time are T_e, T^*_c, ε_1 and ε_2.

Although the inner and outer surfaces of the mould may have different values for the emissivity, due to possible sedimentation on the inner surface, the most reasonable assumption is that $\varepsilon_0 = \varepsilon_1 = \varepsilon_2$. Nevertheless it is instructive to consider the effect of varying the parameter α, $\alpha = \varepsilon_1/\varepsilon_0 = \varepsilon_2/\varepsilon_0$, as this clearly shows the importance of the gap, because with $\alpha = 1$ we have the previously mentioned reasonable assumption and letting $\alpha \to \infty$ corresponds to the absence of a gap. Figure 1 shows this variation for the particular case of a 7 t ingot when $T_e = 20°C$ and $T^*_c = 100°C$; the time of complete solidification when $\alpha = 1$ is approximately 23% longer than the asymptotic value as $\alpha \to \infty$ and illustrates the important qualitative effect of including the gap. Table 1 displays the times of solidification with $\alpha = 1$ for the various ingots, in particular the mould predicts $t_s = 197$ min which may be compared with the observed time of 195 min*. It should be noted that $\alpha = 1$ is not chosen as a "fudge factor", rather $\alpha = 1$ seems the most appropriate value and hence the model appears to be quantitatively correct. This confirms the claim of Massey and Sheridan [8] that pure radiation is the correct model; the importance of conduction in the gap could be assessed by solving the integro-differential equation governing combined radiation and conduction in the gap as outlined by Hobson [10], but for the sake of this paper the noted agreement is accepted as sufficient confirmation.

Having established the major point about the importance of the gap and accuracy of the model it is

Table 1. Ingot sizes and the theoretical times of complete solidification

Ingot size (t)	Equivalent radii for cylindrical model	t_s (min)
7	$a = 0.37$ m $b = 0.43$ m	157
8	$a = 0.385$ m $b = 0.458$ m	165
10	$a = 0.432$ m $b = 0.554$,	197

Table 2. Theoretical times, in minutes, for solidification of a 7 t ingot for various environment and preheat temperatures

T^*_c	T_e 20°C	100°C
20°C	155.0	155.4
100°C	157.0	157.4
200°C	161.8	162.2

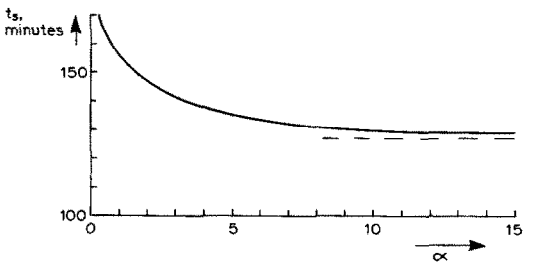

FIG. 1. Variation of solidification time, t_s, of a 7 t ingot with the parameter α.

*Private communication from British Steel Industry, Scunthorpe.

interesting to consider variations in the other parameters which may affect the total time of solidification which itself has an effect on the physical properties of the material. The environmental temperature T_e can vary for different ingots depending on whether they stand alone in a cool part of the foundry or adjacent to other hot moulds; Table 2 shows that the times are not changed by any appreciable amount. Table 2 also shows that preheating the mould has only a marginal effect on t_s; another expected result because the rate of solidification is dependent on the amount of heat which needs to be transferred through the mould and the majority of this is the latent heat liberated as the metal solidifies.

REFERENCES

1. D. N. de G. Allen and R. T. Severn, The application of the relaxation method to the solution of non-elliptic partial differential equation, *Q. J. Mech. Appl. Math.* **15,** 53 (1962).
2. G. Poots, On the application of integral-methods to the solution of problems involving the solidification of liquids initially at fusion temperature, *Int. J. Heat Mass Transfer* **5,** 525 (1962).
3. D. S. Riley, F. T. Smith and G. Poots, The inward solidification of spheres and cylinders, *Int. J. Heat Mass Transfer* **17,** 1507 (1974).
4. K. Stewartson and R. T. Waechter, On Stefan's problem for spheres, *Proc. R. Soc.* **348,** 415 (1976).
5. J. S. Goodling and M. S. Khader, Inward solidification with radiation-convection boundary conditions, *J. Heat Transfer* **96,** 114 (1974).
6. R. W. Ruddle, The solidification of castings, Institute of Metals Monograph 7 (1950).
7. P. M. Beckett, The effects of a containing sleeve on the solidification of a cylindrical bar. *Mech. Res. Comm.* **6,** 129 (1979).
8. I. D. Massey and A. T. Sheridan, Theoretical predictions of earliest rolling times and solidification times of ingots, *J. Iron Steel Inst.* **209,** 111–351 (1971).
9. D. R. Hartree and J. R. Womersley, A method for the numerical or mechanical solution of certain types of partial differential equations, *Proc. R. Soc.* **161A,** 353 (1937).
10. N. Hobson, A theoretical investigation into the solidification of a cylindrical steel ingot, M.Sc. Thesis, University of Hull (1977).

L'EFFET DU RETRAIT SUR LA VITESSE DE SOLIDIFICATION D'UN LINGOT CYLINDRIQUE

Résumé — Lors du refroidissement, la partie solidifiée d'un lingot se contracte et il en résulte un petit espace entre elle et le moule. Des équations sont établies qui tiennent compte de l'existence d'un tel espace dans lequel le transfert thermique est supposé purement radiatif. Des solutions numériques montrent que la présence de cet écart augmente le temps de la solidification complète et rendent compte des sousestimations antérieures de ce temps, en accord avec les observations dans les aciéries.

DER EINFLUß DES SCHRUMPFENS AUF DIE ERSTARRUNGSGESCHWINDIGKEIT EINES ZYLINDRISCHEN BLOCKS

Zusammenfassung—Durch Abkühlung zieht sich der erstarrte Teil eines Blocks zusammen und erzeugt einen schmalen Spalt zwischen dem Block und seiner Kokille. Es werden Gleichungen aufgestellt, welche die Existenz eines solchen Spalts berücksichtigen, in dem Wärmeübertragung ausschließlich durch Strahlung angenommen wird. Numerische Lösungen zeigen, daß das Vorhandensein des Spalts die Zeit bis zum vollständigen Erstarren vergrößert und erklären frühere—im Vergleich zu Beobachtungen in der Stahlindustrie—zu niedrige Schätzungen der Erstarrungsdauer.

ВЛИЯНИЕ УСАДКИ НА ИНТЕНСИВНОСТЬ ЗАТВЕРДЕВАНИЯ ЦИЛИНДРИЧЕСКОЙ ОТЛИВКИ

Аннотация — По мере охлаждения отливки затвердевающая её часть сжимается, и между отливкой и формой образуется небольшой зазор. С учётом наличия такого зазора сформулированы уравнения, в которых предполагается, что перенос тепла осуществляется только излучением. С помощью численных решений показано, что зазор увеличивает время полного затвердевания отливки, и дано объяснение полученным ранее заниженным значениям этого времени по сравнению с данными сталелитейной промышленности.